

SPLITTING A CONTRACTION OF A SIMPLE CURVE TRAVERSED m TIMES

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ABSTRACT. Suppose that M is a 2-dimensional oriented Riemannian manifold, and let γ be a simple closed curve on M . Let $m\gamma$ denote the curve formed by tracing γ m times. We prove that if $m\gamma$ is contractible through curves of length less than L , then γ is contractible through curves of length less than L .

In the last section we state several open questions about controlling length and the number of self-intersections in homotopies of curves on Riemannian surfaces.

1. INTRODUCTION

The main result of this article extends Theorem 1.2 from [CL]:

Theorem 1.1 ([CL] Theorem 1.2). *Suppose M is a 2-dimensional orientable Riemannian manifold (possibly with boundary), γ is a closed curve on M , and 2γ is the closed curve formed by traversing γ twice. If 2γ can be contracted through curves of length less than L , then γ can also be contracted through curves of length less than L .*

In this article, we extend this theorem in the following way:

Theorem 1.2. *Suppose M is a 2-dimensional orientable Riemannian manifold (possibly with boundary), γ is a simple closed curve on M , and $m\gamma$ is the curve formed by traversing γ m times (with m an integer). If $m\gamma$ can be contracted through curves of length less than L , then γ can also be contracted through curves of length less than L .*

This new theorem extends the original theorem in that it considers m -iterates of γ for arbitrary m , however, it only pertains to *simple* closed curves. Given a contraction of $m\gamma$, we give an explicit algorithm for constructing a homotopy of γ . Moreover, the contraction that we obtain is in fact an isotopy (except for the final constant curve).

The general structure of proof builds on the ideas in [CL]. As in that article, we prove the theorem by constructing a graph, and then examining the degree of that graph. In particular, we look at various ways of cutting each curve in our homotopy at self-intersection points and rejoining them to form collections of simple closed

curves. These form the vertices of our graph. We join a pair of vertices by an edge if there are isotopies between the collections of simple closed curves represented by the vertices.

We then argue that we can start at a vertex which corresponds to m copies of γ and find a path in this graph to a point where the number of simple closed curves decreases. This implies that we have a contraction of one of the original curves which is isotopic to γ . This portion of the argument, as in [CL], relies on Euler's handshaking lemma from his famous solution of the Königsberg's bridges problem.

This result can be viewed as an effective version of the fact that closed orientable 2-dimensional manifolds have no torsion; we cannot find a simple closed curve which is more difficult to contract than the curve formed by traversing it m times.

In the last section we present some related open problems.

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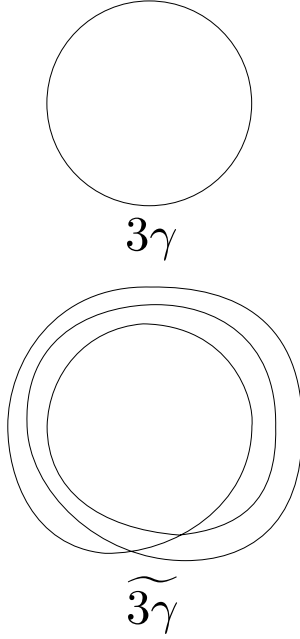
2. PRELIMINARIES

We begin by perturbing the given contraction of $m\gamma$ so that the new contraction still consists of curves of length less than L , but that the initial curve has $m - 1$ self-intersections, as in Figure 1. Furthermore, we can perform this perturbation so that, if we cut this curve at every self-intersection, we obtain m simple curves, each of which is isotopic of γ through curves of length less than L (see Figure 8). We may assume that H has already been perturbed in this way.

As in [CL], we use a parametric form of Thom's Multijet Transversality Theorem as well as other standard methods (see [A], [D], [Bru] and [GG]) to perturb our homotopy H to obtain a homotopy \tilde{H} such that

- (1) $\tilde{H}(0) = H(0)$
- (2) $\tilde{H}(1)$ is a closed curve which lies in a small disc possessing the property that any simple closed curve in this disc of length less than L can be contracted in that disc through curves of length less than L .
- (3) The length of $\tilde{H}(t)$ is less than L for every t .
- (4) For all but finitely many $t \in [0, 1]$, $H(t)$ is an immersed curve with finitely many transverse self-intersections and no triple points.
- (5) There exists a finite number of times $t_0 < \cdots < t_n \in [0, 1]$ with $t_0 = 0$ and $t_n = 1$ such that exactly one Reidemeister move occurs between t_i and t_{i+1} for every $1 \leq i < n$. For every point t_i , t_i satisfies (4).

The Reidemeister moves that we refer to are so named because of their obvious analogy to the corresponding moves in knot theory (see [K]). There are three types of them, and they are shown in Figure 2. Note that since $H(0)$ contains self-intersections

FIGURE 1. Perturbation of 3γ to form $\widetilde{3\gamma}$

and $H(1)$ is a simple closed curve, there must be at least one Reidemeister move; as such, $n \geq 1$. For the remainder of this article, we will assume that our homotopy has already been put into this form.

3. DEFINITION OF THE GRAPH Γ

For each t_i , consider $H(t_i)$. Let the self-intersections of $H(t_i)$ be s_1, \dots, s_j . For each s_i , we can find a small open ball B centered at s_i so that $H(t_i) \cap B$ consists of two arcs that intersect transversally. If we choose an open ball B' centered at s_i which is much smaller than B , then $H(t_i) \cap B \setminus B'$ consists of four open arcs which have no self-intersections. There are exactly two ways of connecting these four arcs inside of B' to obtain two arcs which do not self-intersect. These are shown in Figure 3. Furthermore, if we choose B' to be small enough, then we can perform either of these surgeries so that the length of the resulting collection of arcs does not exceed L . As a result of these surgeries we will obtain a collection of at most $j + 1$ closed curves.

As in [CL], we will define a graph Γ whose properties will allow us to prove the theorem.

Vertices We begin by defining the vertices. We will define $n + 1$ sets of vertices; their disjoint union will form the set of vertices of Γ . These sets will be denoted

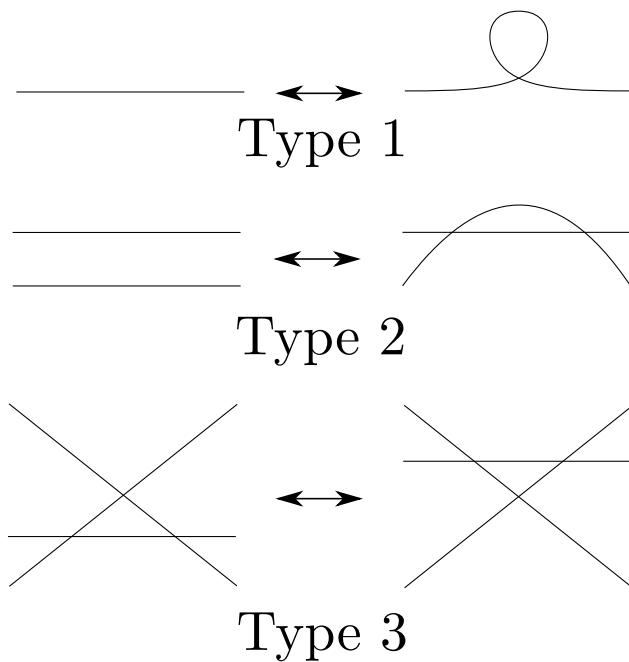


FIGURE 2. The 3 types of Reidemeister moves

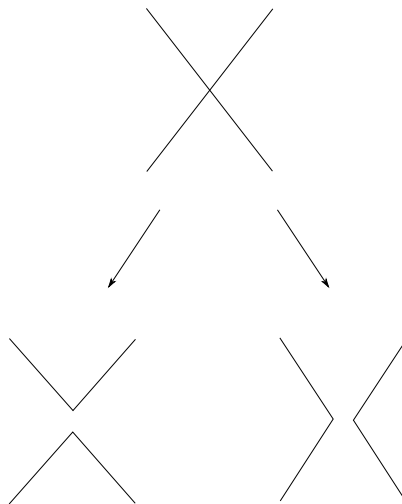


FIGURE 3. Possible resolutions of a self-intersection

as V_0, \dots, V_n . For every i with $0 \leq i \leq n$, consider all collections of arcs formed by cutting every self-intersection of $H(t_i)$ in each of the two possible ways shown in Figure 3. If there are j self-intersections, then there are 2^j such collections. For each

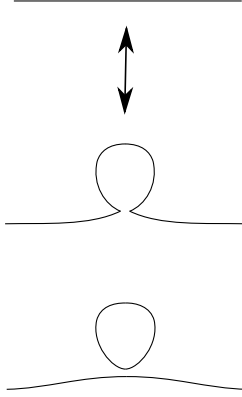


FIGURE 4. Possible edges resulting from a Type 1 move

collection of simple closed curves obtained from a cutting, we add a vertex to V_i . As previously stated, the vertices of Γ are $\sqcup_{i=0}^n V_i$.

Edges We now proceed to adding edges. Every edge in Γ is either between a vertex v in V_i and a vertex w in V_{i+1} , or two vertices v and w in V_i . Fix a vertex v in V_i , and let R_1 be the Reidemeister move between t_{i-1} and t_i (if $i = 0$, then we skip this step). We now add an edge from v to a vertex w based on the type of move which R_1 corresponds to. Let C_v and C_w be the collections of simple closed curves which v and w respectively represent.

If R_1 is of Type 1, and if C_1 and C_2 locally correspond to diagrams which are linked in Figure 4, then we add an edge between v and w .

If R_1 is of Type 2, and if C_1 and C_2 locally correspond to diagrams which are linked in Figure 5, then we add an edge between v and w .

If R_1 is of Type 3, and if C_1 and C_2 locally correspond to diagrams which are linked in Figure 6, then we add an edge between v and w .

Let R_2 be the Reidemeister move between t_i and t_{i+1} (if $i = n$, then we skip this step). We now follow the identical procedure as we followed with R_1 to add edges to Γ . Note that if we add an edge between v and a vertex w in the first step, and we add an edge between v and w in this second step, then Γ has two edges between v and w .

4. PROOF OF THEOREM 1.2

Lemma 4.1. *If vertices v and w in Γ that correspond to collections of curves C_v and C_w are connected by an edge, then there is a bijection between the curves in C_v and the curves in C_w such that there is an isotopy between each pair of curves that are obtained from this bijection.*

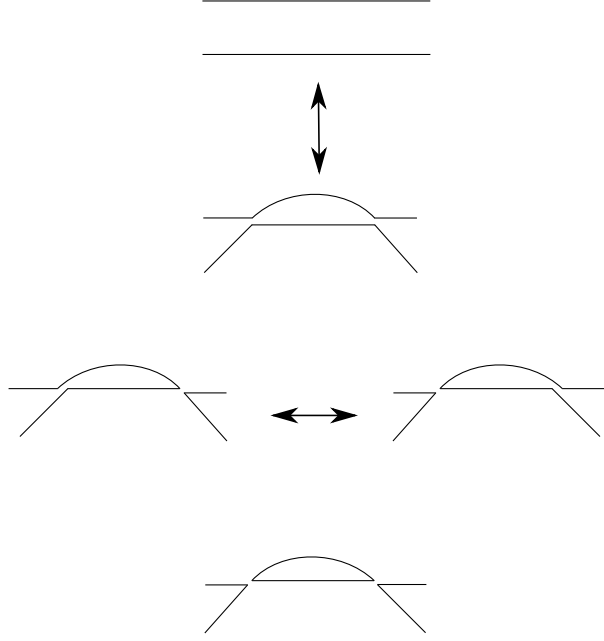


FIGURE 5. Possible edges resulting from a Type 2 move

Proof. This is proved on a case-by-case basis. Each edge e is added as a result of a Reidemeister move R . In every case, we observe that the number of simple closed curves does not change. The fact that the curves are isotopic in each case is clear from Figures 4, 5 and 6 (see the proof of Theorem 1.2 in [CL]). \square

Lemma 4.2. *For every vertex v in Γ , if v has odd degree, then at least one of the following properties is true:*

- (1) v lies in V_0 .
- (2) v lies in V_n .
- (3) *There is a simple closed curve in the collection of curves which corresponds to v that is contractible through simple closed curves of length less than L .*

Proof. Assume that there is a vertex v that has none of these properties, but which has odd degree. We have that $v \in V_i$ with $i \neq 0$ and $i \neq n$. Let R_1 be the Reidemeister move between t_{i-1} and t_i , and let R_2 be the Reidemeister move between t_i and t_{i+1} . The edges with endpoint v are produced from R_1 and from R_2 . Looking at Figures 4, 5, and 6, we see that in each case R_1 produces either 0, 1, or 3 edges. If it produces 0 edges, then v must locally look like one of the diagrams in Figure 7 depending on the type of R_1 . In each case, the simple closed curve shown in each diagram can be contracted to a constant curve within a very small disc; as such, it

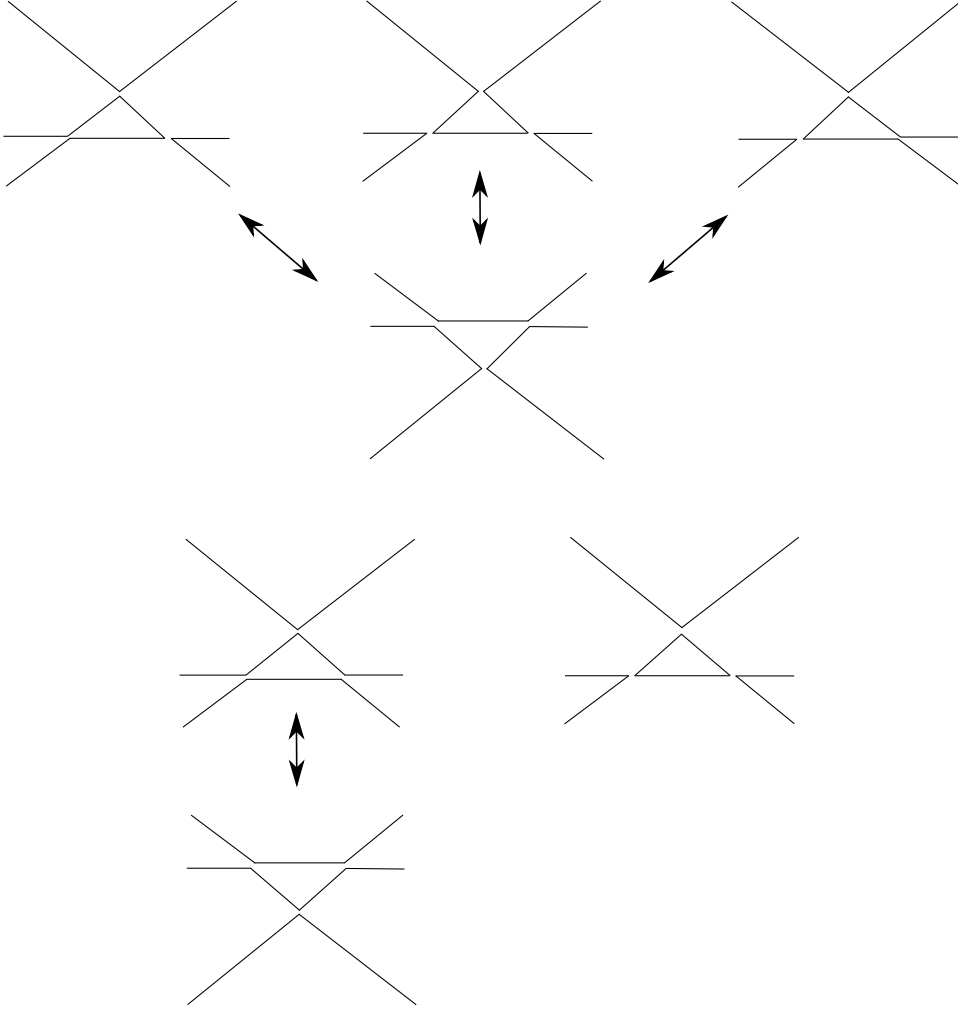


FIGURE 6. Possible edges resulting from a Type 3 move

can be contracted through curves of length less than L . Since v does not have this property, R_1 must add 1 or 3 edges to v .

The identical analysis for R_2 demonstrates that it too contributes 1 or 3 edges to v . As such, the number of edges with endpoint v is 2, 4, or 6, proving that it has even degree.

□

We will look at a certain subgraph Γ' of Γ . This subgraph is defined as the connected component of Γ which contains a particular vertex $v^* \in V_0$. $H(0)$ has $m - 1$ self-intersections; if we make the correct choice for resolving each vertex, then

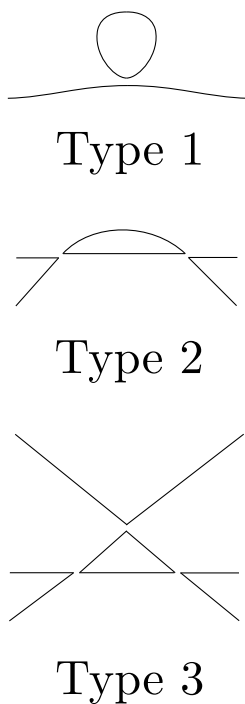


FIGURE 7. The possible diagrams that correspond to vertices with 0 edges

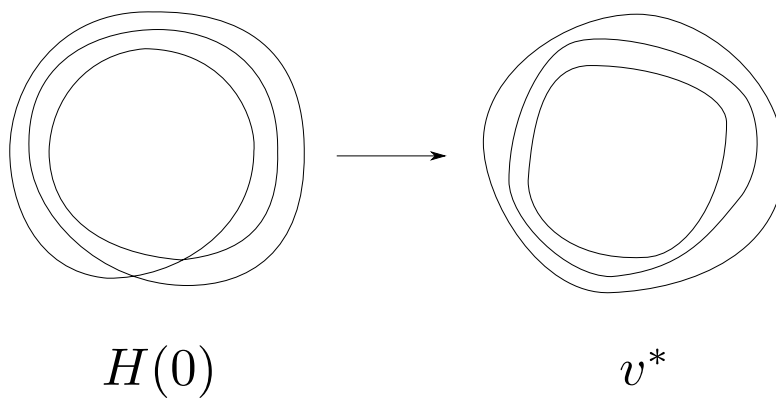


FIGURE 8. The redrawing which corresponds to the first vertex in our path

we obtain m simple closed curves, each of which is isotopic to γ through curves of length less than L . v^* is the vertex in Γ that corresponds to this collection of simple closed curves. See Figure 8 for an example of such a collection of curves for $m = 3$.

Proposition 4.3. *For any vertex w in Γ' , for any simple closed curve α in the collection of curves which corresponds to w , α is isotopic to γ through curves of length less than L .*

Proof. Since w is in Γ' , there is a path in Γ from v^* to w . Lemma 4.1 implies that α is isotopic to one of the simple closed curves in the collection corresponding to v^* through curves of length less than L . Each of the simple closed curves in this collection is isotopic to γ through curves of length less than L , completing the proof. \square

We can now prove Theorem 1.2.

Proof of Theorem 1.2. Suppose that v^* is of even degree. Let R correspond to the Reidemeister move between t_0 and t_1 . The only edges that have v^* as an endpoint come from R , and so R must add 0 edges to v^* . From the discussion in the proof of Lemma 4.2 and Figure 7, we have that one of the simple closed curves in the collection corresponding to v^* is contractible through simple closed curves of length less than L . Each of the curves in this collection is isotopic to γ through curves of length less than L , completing the proof.

Suppose now that v^* is of odd degree. By Euler's handshaking lemma, Γ' must contain another vertex $w \neq v^*$ that also has odd degree. By Lemma 4.2, $w \in V_0$, $w \in V_n$, or one of the simple closed curves in the collection that corresponds to w is contractible through curves of length less than L . We proceed on a case-by-case basis:

Case 1: $w \in V_0$ This case cannot occur. Assume that $w \in V_0$. Then, by Lemma 4.1, w and v^* must correspond to collections consisting of the same number of simple closed curves. However, since $H(0)$ has $m-1$ self-intersections, there is exactly one resolution of these self-intersections which produces m simple closed curves, and this resolution corresponds to v^* . Hence, w must correspond to a collection of strictly less than m simple closed curves, and so we obtain a contradiction.

Case 2: $w \in V_n$ Choose a simple closed curve α that is in the collection of simple closed curves which corresponds to w . By Proposition 4.3, α is isotopic to γ through curves of length less than L . Since α is composed of arcs of $H(1)$ and $H(1)$ lies in a disc possessing the property that any simple closed curve of length less than L can be contracted through simple closed curves of length less than L , α is contractible through simple closed curves of length less than L .

Case 3: $w \notin V_0$ and $w \notin V_n$ In this case, by Lemma 4.2, there is a simple closed curve α which is in the collection of simple closed curves that corresponds to w , and which has the property that it can be contracted through simple closed curves of length less than L . By Proposition 4.3, α is isotopic through curves of length less than L to γ , completing the proof. \square

5. OPEN PROBLEMS

Several natural open questions arise from the problems considered in [CL], in [CL2], and in this paper.

The first one is whether Theorem 1.2 holds without the assumption that γ is simple.

Conjecture 1 Suppose M is a 2-dimensional orientable Riemannian manifold, γ is a closed curve on M , and $m\gamma$ is the curve formed by traversing γ m times. If $m\gamma$ can be contracted through curves of length less than L , then γ can also be contracted through curves of length less than L .

One may try to approach this problem using the methods of this paper by perturbing $m\gamma$ and cutting it at the self-intersections points to obtain $m - 1$ copies of γ . One would then like to define a graph of resolutions similar to the ones constructed in this paper, in [CL], or in [CL2]. Unfortunately, the most straightforward choices lead to graphs with many vertices of odd degree. However, it seems to us that one may still be able to obtain a contraction of γ by studying some finer properties of these graphs and, possibly, by using some additional surgeries (cf. the surgeries on the homotopies of curves in [CL2]).

Defining such a graph and studying its properties may lead to a generalization of Theorem 1.1 from [CL]. In [CL] the authors proved that, given a contraction of a simple closed curve through curves of length less than L , one can find a contraction through simple curves satisfying the same length bound. This leads to the following conjecture.

Conjecture 2 Let γ be a closed curve with k self-intersections, and suppose that γ can be contracted through curves of length less than L . γ can then be contracted through curves of length less than L with at most k self-intersections.

If Conjecture 2 holds, then it implies an a priori somewhat stronger statement that the contraction of γ can be chosen so that the number of self-intersections decreases monotonically (in other words, intersections are only destroyed in the homotopy and never created).

Finally, it is of interest whether parametric versions of these results hold.

Question 3 Let $\{\gamma^s\}$ be a k -parameter family of closed curves on an orientable Riemannian surface and suppose that the family $\{2\gamma^s\}$ can be continuously deformed to a family of points through curves of length less than L . Does there exist a similar deformation of $\{\gamma^s\}$ through curves of length less than L ?

This problem is closely related to a question of N. Hingston and H.-B. Rademacher (see [HR] and [BM]) about min-max levels of multiples of homology classes of the loop space of a Riemannian sphere.

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